7.2

2) Consider the function h defined by

$$h(x) := \begin{cases} x+1 & , x \in [0,1] \cap \mathbb{Q} \\ 0 & , x \in [0,1] \cap (\mathbb{R} \setminus \mathbb{Q}) \end{cases}$$

Show that h is not Riemann integrable

Pf: We use the Cauchy Criterion to show that $h \notin \mathcal{R}[0,1]$. Take $\varepsilon_0 = \frac{1}{2}$

Given $\delta > 0$,

consider tagged partitions $\dot{\mathcal{P}}, \dot{\mathcal{Q}}$ such that $\|\dot{\mathcal{P}}\|, \|\dot{\mathcal{Q}}\| < \delta$ but all the tags p_i for $\dot{\mathcal{P}}$ are rational, while all the tags q_j for $\dot{\mathcal{Q}}$ are irrational.

Note, we can always do this because the norm of the partition is independent of choice of tags.

Then note that since $h(p_i) = p_i + 1 \geq 1$, we have

$$S(h; \dot{\mathcal{P}}) = \sum_{i=1}^{n} h(p_i)(x_{i+1} - x_i) \geq \sum_{i=1}^{n} (x_{i+1} - x_i) = 1, \text{ but}$$

$$S(h; \dot{\mathcal{Q}}) = \sum_{j=1}^{m} h(q_j)(x_{j+1} - x_j) = 0, \text{ so}$$

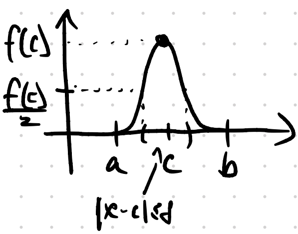
$$|S(h; \dot{\mathcal{P}}) - S(h; \dot{\mathcal{Q}})| = 1 \geq \varepsilon_0 > 0.$$

So $h \notin \mathcal{R}[0,1]$.

8) Suppose that f is continuous on $[a,b]$, that $f(x) \geq 0$ for all $x \in [a,b]$ and that $\int_a^b f = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$.

Pf: Suppose for the sake of contradiction that there is a $c \in [a,b]$ such that $f(c) > 0$.

We first consider the case where $c \in (a,b)$. Then by continuity of f, there exists a $\delta > 0$ such that for all $|x-c| \leq \delta$, then $f(x) > \frac{1}{2} f(c)$.



Then we calculate that

$$\int_a^b f(x) \geq \int_{c-\delta}^{c+\delta} f(x) > \int_{c-\delta}^{c+\delta} \tfrac{1}{2} f(c) = \tfrac{1}{2} f(c)(2\delta) > 0.$$

which contradicts the assumption that $\int_a^b f = 0$.

Now suppose c is the endpoint a. Then again by continuity, $\exists \delta > 0$ s.t. for all $x \in [a, a+\delta]$, then $f(x) > \frac{1}{2} f(c)$. Then the same calculation shows

$$\int_a^b f(x) \geq \int_a^{a+\delta} f(x) > \int_a^{a+\delta} \tfrac{1}{2} f(c) = \tfrac{1}{2} f(c)(\delta) > 0$$ which is again a contradiction.

When $c = b$ a similar argument applies. //

9) Show that the continuity hypothesis in the preceeding exercise cannot be dropped.

Pf: Let $h(x) = \begin{cases} 1, & x=0 \\ 0, & x \neq 0. \end{cases}$ Then h is discontinuous at 0,

but we can also show that $\int_0^1 h = 0$. Let $\varepsilon > 0$ be given. Take $\delta = \frac{\varepsilon}{2}$.

Then consider any tagged partition $\dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta$. We have 2 cases:

Case 1: first tag $t_1 = 0$. Then since $t_j \neq 0$ for all $j > 1$, $h(t_j) = 0$ and

Then $|S(h;\dot{\mathcal{P}})| = \left|\sum_{i=1}^{n} h(t_i)(x_{i+1} - x_i)\right| = |h(t_1)(x_2 - x_1)|$

$$= |x_2 - x_1| < \delta = \frac{\varepsilon}{2} < \varepsilon.$$

Case 2: none of the tags are 0. Then $h(t_i) = 0$ for all i, and

$$|S(h;\dot{\mathcal{P}})| = \left|\sum_{i=1}^{n} h(t_i)(x_{i+1} - x_i)\right| = 0 < \varepsilon.$$

So $\int_0^1 h = 0$.

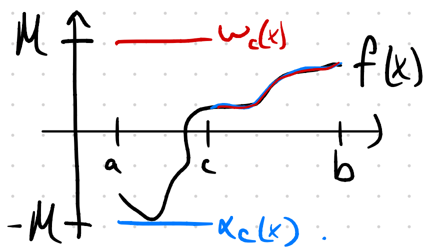
Alternatively take h to be Thomae's function and proceed as in Example 7.1.7 as in the textbook. //

11) If f is bounded by M on $[a,b]$ and if the restriction of f to every interval $[c,b]$ where $c \in (a,b)$ is Riemann integrable, show that $f \in \mathcal{R}[a,b]$ and that $\int_c^b f \to \int_a^b f$ as $c \to a^+$.

Pf: We'll use the Squeeze Theorem 7.2.3 in the textbook. Let $\varepsilon > 0$ be given. In the notation of that theorem and following the hint provided, take

$$\alpha_c(x) = \begin{cases} -M, & x \in [a,c) \\ f(x), & x \in [c,b]. \end{cases}$$

$$\omega_c(x) = \begin{cases} M, & x \in [a,c) \\ f(x), & x \in [c,b]. \end{cases}$$



Then by the Additivity Theorem (7.2.9), both $\alpha_c, \omega_c \in \mathcal{R}[a,b]$ since constant functions $-M, M \in \mathcal{R}[a,c]$, and $f(x) \in \mathcal{R}[c,b]$ by assumption. Moreover, we have

$\alpha_c(x) \leq f(x) \leq \omega_c(x)$ for all $x \in [a,b]$ and

$$\int_a^b (\omega_c - \alpha_c) = \int_a^c 2M + \int_c^b (f(x) - f(x)) = 2M(c-a).$$

Taking c close enough to a so that $c - a < \frac{\varepsilon}{2M}$, we get

$\int_a^b (\omega_c - \alpha_c) < \varepsilon$. Then by the Squeeze Theorem, we can conclude that $f \in \mathcal{R}[a,b]$. Moreover, for this c, we have

$$\left| \int_a^b f - \int_c^b f \right| = \left| \int_a^c f \right| \leq M(c-a) < \frac{\varepsilon}{2} < \varepsilon. \text{ So } \int_c^b f \to \int_a^b f \text{ as } c \to a^+.$$

//.

12) Show that $g(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \in (0,1] \\ 0, & x = 0 \end{cases}$ belongs to $\mathcal{R}[0,1]$.

Pf: We can use the previous problem. Note that

$|g(x)| \leq 1$ for all $x \in [0,1]$. So g is bounded on $[0,1]$.

Moreover, since g is continuous on every interval $[c,1]$ for all $c \in (0,1)$, $g \in \mathcal{R}[c,1]$ (Thm 7.2.7). Then by the previous problem, $g \in \mathcal{R}[0,1]$.

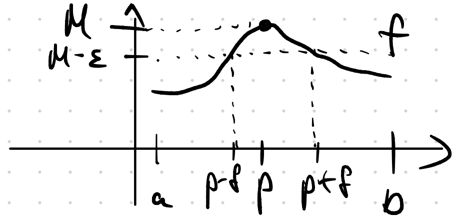
18) Let f be continuous on $[a,b]$, let $f(x) \geq 0$ for $x \in [a,b]$, and let $M_n := \left(\int_a^b f^n\right)^{1/n}$. Show that

$$\lim M_n = \sup\{f(x) : x \in [a,b]\}.$$

Pf: Let $M := \sup\{f(x) : x \in [a,b]\}$. Since f is continuous and $[a,b]$ is compact, f achieves M at some point, say $p \in [a,b]$. ie $f(p) = M$. By continuity of M, $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t.

for all $x \in (p-\delta, p+\delta)$,

$$M - \varepsilon \leq f(x) \leq M.$$



Then integrating, we have

$$(M-\varepsilon)^n 2\delta \leq \int_{p-\delta}^{p+\delta} f^n \leq \int_a^b f^n \leq M^n (b-a).$$

Then taking nth root, we have

$$(M-\varepsilon) 2\delta^{1/n} \leq \left(\int_a^b f^n\right)^{1/n} \leq M(b-a)^{1/n}.$$

Note that for any $r > 0$, $\lim_{n \to \infty} r^{1/n} = 1$ since $\lim_{n\to\infty} \log(r^{\frac{1}{n}}) = \lim_{n\to\infty} \frac{1}{n}\log(r) = 0$ and using continuity of log on $\mathbb{R}_+$ and by taking exponential.

So taking limits, we get

$$M-\varepsilon \leq \lim_{n\to\infty} M_n \leq M.$$

Since this is true for arbitrary $\varepsilon>0$, we can conclude $\lim_{n\to\infty} M_n = M$.